Notes on Homotopy and Obstruction Theory

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Abstract

A list of definitions and theorems in preparation for my Algebraic Topology final. This roughly covers Hatcher, Ch. 4: Homotopy, as well as the section in Fuchs, Fomenko on Obstruction Theory. Any typos and mistakes are my own - kindly direct them to my inbox.

Contents

1 Homotopy Groups

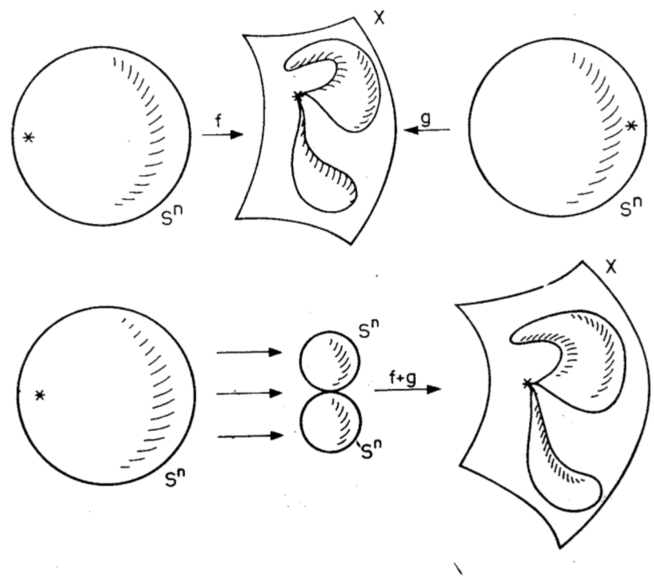
We let I^n denote the n-dimensional cube.

Definition 1.1 (Homotopy Group)**.** Let $n \geq 1$. Define $\pi_n(X, x_0)$ to be the set of homotopy classes of maps $f : (I^n, \partial I^n) \to (X, x_0)$, where homotopies f_t are required to satisfy $f_t(\partial I^n) = x_0$ for all t.

Definition 1.2 (Addition of Homotopies)**.** The addition operation for $\pi_n(X, x_0)$ is given by

$$(f+g)(s_1, s_2, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n), & s \in [0, 1/2] \\ g(2s_1 - 1, s_2, \dots, s_n), & s_1 \in [1/2, 1] \end{cases}.$$

This defines a group operation on π_n.



Proposition 1.3. *$\pi_n(X, x_0)$ is Abelian for $n \geq 2$.*

Definition 1.4. X is n-simple if $\pi_1(X, x_0)$ acts trivially on $\pi_n(X, x_0)$.

1.1 Coverings

Lemma 1.5. *Suppose $p : \tilde{X} \to X$ is a covering. Then, $\pi_1(\tilde{X})$ injects into $\pi_1(X)$, with the index given by the degree of the covering.*

Lemma 1.6. *Suppose $p : \tilde{X} \to X$ is a covering. Then, $p_* : \pi_n(\tilde{X}) \to \pi_n(X)$ is an isomorphism for $n \geq 2$.*

1.2 Relative Homotopy

Definition 1.7. Suppose (X, A, x_0) satisfies $x_0 \in A \subset X$. We can define the n-th relative homotopy group of (X, A, x_0) as the set of homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ where $I_n \supset I^{n-1} = \{x \in I^n : s_n = 0\}$ and $J^{n-1} = \partial I^n \setminus I^{n-1}$, and the homotopy is defined as before.

Proposition 1.8. *There exists a long exact sequence*

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\delta} \pi_{n-1}(A, x_0) \to \dots,$$

where i_* is the induced map by $(A, x_0) \to (X, x_0)$, j_* is the induced map by $(X, x_0, x_0) \to (X, A, x_0)$ and δ is the restriction to the boundary.

1.3 Whitehead's Theorem

Proposition 1.9. *If $f : (X, x_0) \to (Y, y_0)$ is a homotopy equivalence, then $f_{*n} : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism for all n.*

Theorem 1.10 (Whitehead). *Suppose X, Y are CW-complexes and $f : (X, x_0) \to (Y, y_0)$ is a map such that f_* is an isomorphism for all n. Then, f is a homotopy equivalence.*

Theorem 1.11 (Hurewitz). *Suppose that $\pi_i(X) = 0$ for all $i < n$ and $n \geq 2$. Then $\tilde{H}_i(X, \mathbb{Z}) = 0$ for all $i < n$ and $H_n(X; \mathbb{Z}) \cong \pi_n(X)$.*

2 Obstruction Theory