Notes on Homology

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Abstract

A list of definitions and theorems in preparation for my Algebraic Topology midterm. This roughly covers Hatcher, Ch. 2: Homology, with an emphasis on the material from 2.1, 2.2, and 2.B. Any typos and mistakes are my own - kindly direct them to my inbox.

Contents

1 Simplicial and Singular Homology

1.1 Simplicial Complexes

Definition 1.1 (Simplex)**.** The n-simplex(denoted Δ^n) is given by the convex hull of $n+1$ linearly independent vectors in $\mathbb{R}^{n+1}$: given $e_0, \ldots, e_n \in \mathbb{R}^{n+1}$,

$$\Delta^n = \left\{ \sum_{i=0}^{n} \lambda_i e_i : \lambda_i \in [0,1], \sum_{i=0}^{n} \lambda_i = 1 \right\}.$$

We also denote $\Delta^n = [e_0, \ldots, e_n]$.

Definition 1.2 (Face)**.** A face of a simplex $[v_0, \ldots, v_n]$ is given by the convex hull of the remaining vectors upon removing one of them. We denote this $[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$, where v_i is the vector that is removed.

Definition 1.3 (Simplicial Complex)**.** Let V be a finite (nonempty) set. A nonempty subset $S \subset \mathcal{P}(V)$ is a simplicial complex if $\bigcup S = V$ and $A \in S, B \subset A$ implies that $B \in S$.

1.2 Simplicial Homology

Definition 1.4. Let S be a simplicial complex. Define

$$C_n(S) = \{\sum_{i \in I} n_i A_i : A_i \in S, |A_i| = n+1, n_i \in \mathbb{Z}\}.$$

The elements of $C_n(S)$ are called n-chains.

Definition 1.5. We define the boundary homomorphisms, $\partial_n : C_{n+1}(S) \to C_n(S)$ given by

$$\partial_n(A) = \sum_{j=0}^{n} (-1)^j [v_{i_0}, \ldots, \widehat{v_{i_j}}, \ldots, v_{i_{n+1}}],$$

where $A \in S$, $|A| = n+2$ and $A = [v_{i_0}, \ldots, v_{i_{n+1}}]$.

Lemma 1.6. $\partial_{n-1} \circ \partial_n \equiv 0$.

Definition 1.7 (Cycles and Boundaries)**.** The cycles and boundaries of $C_n(S)$ are given by the subgroups $Z_n(S) = \ker \partial_{n-1}$ and $B_n(S) = \operatorname{im} \partial_n$, respectively.

Definition 1.8 (Simplicial Homology)**.** We define the n-th simplicial homology group of S as $H_n(S) = Z_n(S)/B_n(S)$.

1.3 Singular Homology

We consider a topological space X.

Definition 1.9 (Singular Simplex)**.** A continuous map $\sigma : \Delta^n \to X$ is a singular simplex.

Definition 1.10 (Singular Chains)**.** We define $C_n(X)$ to be the free group generated by n-simplices:

$$C_n(X) = \{\sum_{i \in I} n_i \sigma_i : n_i \in \mathbb{Z}, I \text{ finite}, \sigma_i \text{ is a singular simplex}\}.$$

Definition 1.11 (Boundary Maps)**.** We define $\partial_n : C_{n+1}(X) \to C(X)$ by

$$\partial_n(\sigma) = \sum_{i=0}^{n+1} (-1)^i \sigma|_{[e_0, \ldots, \widehat{e_i}, \ldots, e_{n+1}]},$$

where we have a canonical identification of $[e_0, \ldots, \widehat{e_i}, \ldots, e_{n+1}]$ with Δ^{n-1} preserving the order of the vertices.

Definition 1.12 (Cycles and Boundaries)**.** As before $Z_n(X) = \ker \partial_{n-1}$, $B_n(X) = \operatorname{im} \partial_n$.

Lemma 1.13. $\partial_{n-1} \circ \partial_n : C_{n+1}(X) \to C_{n-1}(X) = 0$.

Definition 1.14 (Singular Homology)**.** $H_n(X; \mathbb{Z}) = Z_n(X; \mathbb{Z})/B_n(X; \mathbb{Z})$.

Proposition 1.15. *If X is decomposed into its path-connected components X_α, there is an isomorphism of $H_n(X)$ with $\bigoplus_\alpha H_n(X_\alpha)$.*

Proposition 1.16. *If X is a nonempty path-connected topological space, then $H_0(X) \cong \mathbb{Z}$.*

Proposition 1.17. *If $X = \{pt\}$, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) = \mathbb{Z}$.*

Definition 1.18 (Reduced Homology)**.** Define the map $\epsilon : C_0(X) \to \mathbb{Z}$ by $\sum_i n_i \sigma_i \mapsto \sum_i n_i$. Using the sequence

$$\ldots \xrightarrow{\partial_n} C_n(X; \mathbb{Z}) \xrightarrow{\partial_{n-1}} \cdots \to C_0(X; \mathbb{Z}) \xrightarrow{\epsilon} \mathbb{Z},$$

and taking homologies, we obtain the reduced homology groups $\tilde{H}_i(X; \mathbb{Z})$.

1.4 Homotopy Invariance

Definition 1.19 (Chain complex)**.** Let C be an abelian group and $\partial \in \operatorname{End}(C)$. The pair (C, ∂) is a chain complex if $\partial^2 = \partial \circ \partial = 0$. The homology of a chain complex is given by $H(C, \partial) = \ker \partial / \operatorname{im} \partial$.

Definition 1.20 (Gradings)**.** A graded group A can be decomposed as $A = \bigoplus_{n \in \mathbb{Z}} A_n$. A homomorphism $f : A \to B$ is graded if $f(A_n) \subset B_n$. A chain complex (C, ∂) is graded if C and ∂ are graded.

Definition 1.21 (Chain Map)**.** Suppose (C, ∂_C), (D, ∂_D) are two chain complexes. A homomorphism $f : C \to D$ is a chain map if $\partial_D \circ f = f \circ \partial_C$.

Proposition 1.22. *A chain map between chain complexes induces a homomorphism between the homology groups of the two complexes, denoted by $f_* : H(C, \partial_C) \to H(D, \partial D)$ given by $[c] \mapsto [f(c)]$. We also denote this functorially as $H(f)$.*

Some basic properties:

- If $f : X \to Y$, $g : Y \to Z$, then $(g \circ f)_* = g_* \circ f_*$.
- $(\mathrm{id}_X)_* = \mathrm{id}_{H(X)}$.

Definition 1.23 (Chain Homotopy)**.** Two graded chain maps $f, g : C \to D$ are said to be chain homotopic if there exists a homomorphism $\varphi = \bigoplus(\varphi_n : C_n \to D_{n+1})$ satisfying $f - g = \partial_D \circ \varphi + \varphi \circ \partial_C$.

Proposition 1.24. *If f and g are chain homotopic, then $H(f) = H(g)$.*

Proposition 1.25. *If X and Y are topological spaces and $f : X \to Y$ is a continuous map, then it induces a chain map $f_\sharp = C_n(X) \to C_n(Y)$, defined by $f_\sharp(\sigma) = f \circ \sigma$ where $\sigma : \Delta^n \to X$ is a generator of $C_n(X)$.*

Corollary 1.26. *If $f : X \to Y$ is a homeomorphism, then $f_* : H(X) \to H(Y)$ is an isomorphism.*

Definition 1.27. Two maps $f, g : X \to Y$ are said to be homotopic if there exists a continuous map $F : X \times [0, 1] \to Y$ such that $F|_{X \times \{0\}} = f$ and $F|_{X \times \{1\}} = y$.

Definition 1.28 (Homotopy Equivalence)**.** A map $f : X \to Y$ is said to be a homotopy equivalence if there exists $g : Y \to X$ so that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y. If a homotopy equivalence exists between X and Y, then X and Y are homotopy equivalent.

Proposition 1.29. *If $f, g : X \to Y$ are homotopic maps, then $f_\sharp, g_\sharp : C(X) \to C(Y)$ are chain homotopic chain maps.*

Corollary 1.30. *If X and Y are homotopy equivalent topological spaces, then $H_n(X) \cong H_n(Y)$.*

2 Computations and Applications

2.1 Exact Sequences and Excision

Definition 2.1 (Short Exact Sequence)**.** Suppose that L, M, N are abelian groups with the following maps:

$$0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0.$$

The sequence is said to be a short exact sequence if α is a monomorphism, β is an epimorphism, and $\ker \beta = \operatorname{im} \alpha$.

Proposition 2.2. *Suppose that L, M, N are graded chain complexes with a short exact sequence*

$$0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0.$$

This induces a long exact sequence

$$\ldots \xrightarrow{\delta_n} H_n(L) \xrightarrow{H_n(\alpha)} H_n(M) \xrightarrow{H_n(\beta)} H_n(N) \xrightarrow{\delta_{n-1}} H_{n-1}(L) \to \ldots$$

> **Remark 2.3.** The connecting homomorphisms δ_n arise from careful diagram chasing. See the proof in the notes for more details.

Theorem 2.4 (Mayer-Vietoris)**.** *Suppose that $X = A \cup B = int\ A \cup int\ B$. There exists a long exact sequence*

$$\ldots \xrightarrow{\delta_n} H_n(A \cup B) \xrightarrow{f_n} H_n(A) \oplus H_n(B) \xrightarrow{g}_n H_n(X) \xrightarrow{\delta}_{n-1} H_{n-1}(A \cap B) \to \ldots.$$

Proposition 2.5. *For $n > 0$, $H_i(\mathbb{S}^n; \mathbb{Z}) = 0$ if $i \neq 0$ or n. Otherwise $H_0(\mathbb{S}^n) = H_n(\mathbb{S}^n) = \mathbb{Z}$ and the reduced homologies are $\tilde{H}_*(S^n) = \mathbb{Z}_{(n)}$.*

Corollary 2.6. *$\mathbb{S}^n$ and $\mathbb{S}^m$ are homotopy equivalent if and only if $n = m$.*

Corollary 2.7. *$\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$.*

Definition 2.8 (Relative Homology)**.** Suppose $A \subset X$ is a pair of topological spaces. Define $C_n(X, A) = C_n(X)/C_n(A)$ with boundary maps $\partial_{(X,A)} : C_n(X, A) \to C_{n-1}(X, A)$ by $\alpha \mapsto \partial_X(\alpha)$. This gives a chain complex $(C_n(X, A), \partial_{(X,A)})$ with corresponding homology $H_n(X, A; \mathbb{Z})$. This also has a long exact sequence induced from

$$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0.$$

Theorem 2.9 (Excision Principle)**.** *Suppose that $Z \subset A \subset X$ and $\overline{Z} \subset int\ A$. Then, the map $i : (X/Z, A/Z) \hookrightarrow (X, A)$ induces an isomorphism $i_* : H_n(X/Z, A/Z; \mathbb{Z}) \to H_n(X, A; \mathbb{Z})$.*

2.2 Degree

Definition 2.10 (Degree)**.** Suppose that $f : \mathbb{S}^n \to \mathbb{S}^n$ is a continuous map. The induced map $f_* : H_n(\mathbb{S}^n) \to H_n(\mathbb{S}^n)$ which are both isomorphic to $\mathbb{Z}$. Every such map corresponds to multiplication by the integer $f_*(1)$, which we define to be the degree of f, denoted $\deg f$.

Some basic properties:

- $\deg \mathrm{id} = 1$.
- If f is not surjective, $\deg f = 0$.
- If f and g are homotopic, then $\deg f = \deg g$.
- $\deg fg = \deg f \deg g$.
- If f is a reflection, $\deg f = -1$.
- The antipodal map of $\mathbb{S}^n$ has degree $(-1)^{n+1}$.
- If $f : \mathbb{S}^n \to \mathbb{S}^n$ has no fixed points, then $\deg f = (-1)^{n+1}$.

Proposition 2.11. $\deg f = \sum_i \deg f|_{x_i}$.

2.3 CW-Complexes

Definition 2.12 (CW-Complex)**.** The topological space X is a CW-complex if it can be constructed inductively as follows:

1. Start with X^0, a discrete space whose points are regarded as 0-cells.
2. Inductively form an n-skeleton from X^{n-1} by

$$X^n = \left(X^{n-1} \bigsqcup D_i \middle/ \{x \sim \varphi_\alpha(x) : x \in \partial D_\alpha^n\}\right),$$

   where D_α^n are n-discs and $\varphi_\alpha : \partial D_\alpha^n \to X^{n-1}$ are the gluing maps.

Proposition 2.13. *If X is a CW-complex, then*

- $H_k(X^n, X^{n-1}; \mathbb{Z}) \cong \begin{cases} 0 & \text{if } k \neq n \\ \mathbb{Z}^{|\{n\text{-cells}\}|} & \text{if } k = n \end{cases}$
- $H_k(X^n) = 0$ *for all* $k > n$.
- *The embedding* $i : X^n \hookrightarrow X$ *induces an isomorphism* $i_* H_k(X^n) \to H_k(X)$ *if* $k < n$.

2.4 Cellular Homology

Definition 2.14. A pair of spaces (X, A) is called good if

- A is non-empty and closed.
- there is an open set $B \subset X$ containing A such that A is a deformation retract of B; there is a map $F : B \times [0,1] \to B$ with $F(b,0) = b$, $F(b,1) \in A$ for $b \in B$ and $F(a,1) = a$ for $a \in A$.

Theorem 2.15. *For a good pair (X, A), the quotient map $q : (X, A) \to (X/A, A/A)$ induces an isomorphism*

$$H(q) : H_i(X, A) \to H_i(X/A, A/A) \cong \tilde{H}_i(X/A) \forall i.$$

Definition 2.16 (CW-Homology)**.** Define $C_n^{CW}(X) = H_n(X^n, X^{n-1}; \mathbb{Z})$ with boundary map $d_{n-1} : C_n^{CW}(X) \to C_{n-1}^{CW}(X)$ defined by the composition of morphisms in the diagram

$$H_n(X^n, X^{n-1}; \mathbb{Z}) \xrightarrow{\delta}_{n-1} H_{n-1}(X^{n-1}; \mathbb{Z}) \xrightarrow{\gamma}_{n-1} H_{n-1}(X^{n-1}, X^{n-2})$$

where δ_{n-1} is the connecting morphism and γ_{n-1} is the induced by the embedding.

Proposition 2.17. *$(C_*^{CW}(X), d_*)$ is a chain complex; $d_{n-1} \circ d_n = 0$.*

Proposition 2.18. *$H_*(C_*^{CW}, d_*) \cong H_*(X; \mathbb{Z})$, where the second H_* is the singular homology.*

Theorem 2.19 (Cellular Boundary Formula)**.** *Let X be a CW-complex where the n-th skeleton X^n is given by $X^n = X^{n-1} \cup_\varphi \bigcup_{\alpha \in A} e_\alpha^n$ and $X^{n-1} = X^{n-2} \cup_\varphi \bigcup_{\beta \in B} e_\beta^{n-1}$. The boundary of a cell is*

$$d_{n-1}(e_\alpha^n) = \sum_{\beta \in B} d_{\alpha\beta} e_\beta^{n-1},$$

where $d_{\alpha\beta}$ is the degree of the map $S_\alpha^{n-1} \to X^{n-1} \to S_\beta^{n-1}$ that is the decomposition of the gluing map of e_α^n with the quotient map collapsing $X^{n-1} - e_\beta^{n-1}$ to a point.

2.5 Euler Characteristic

Definition 2.20 (Euler Characteristic)**.** Suppose that X is a finite CW-complex with c_n n-cells. Then $\chi(X) = \sum_{i=0}^{\infty} (-1)^i c_i \in \mathbb{Z}$ is called the Euler Characteristic.

Proposition 2.21. *If X is a finite CW-complex, then*

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} H_i(X; \mathbb{Z}) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} H_i(X; \mathbb{Q}).$$

The numbers $\operatorname{rank} H_i(X; \mathbb{Z})$ *are called the Betti numbers.*