Notes on Cohomology

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Abstract

A list of definitions and theorems in preparation for my Algebraic Topology final. This roughly covers Hatcher, Ch. 3: Cohomology, with an emphasis on the material from 3.1, 3.2, and 3.3. Any typos and mistakes are my own - kindly direct them to my inbox.

Contents

1 Cohomology Groups

Definition 1.1 (Cochain Complex)**.** Suppose $(\bigoplus_n C_n, \partial)$ is a graded chain complex. The dual cochain complex is defined as

$$C^n : \mathrm{Hom}(C_n, \mathbb{Z}) = \{f : C_n \to \mathbb{Z} : f \text{ is a homomorphism}\},$$

with coboundary $\delta : C^n \to C^{n+1}$ given by

$$C^{n+1} \ni (\delta_n f)(c) := f(\partial_n c) \in \mathbb{Z}.$$

> **Remark 1.2.** It is easy to check that $\delta^2 = 0$.

Definition 1.3 (Cohomology)**.** Given a cochain complex $(C^* = \bigoplus C^n, \delta = \bigoplus \delta_n)$, we define the cohomology

$$H^n(C^*) := \mathrm{Ker}\, \delta_n / \mathrm{Im}\, \delta_{n-1}.$$

Definition 1.4. Suppose X is a topological space. Apply the algebra to the space to obtain $(\bigoplus C_n(X), \partial = \bigoplus \partial_n)$, the singular chain complex associated to X. Dualizing, we obtain a cochain complex $(C^*(X), \delta)$ and cohomology $H^*(X; \mathbb{Z})$.

1.1 The Universal Coefficient Theorem

Definition 1.5 (Ext)**.** Roughly, $\mathrm{Ext}(H_{n-1}(C), G)$ is the group of Abelian extensions of G by $H_{n-1}(C)$. It has the following properties:

- $\mathrm{Ext}(H \oplus H', G) = \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$,
- $\mathrm{Ext}(H, G) = 0$ if H is free.
- $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) = G/nG$.

Theorem 1.6 (Universal Coefficient Theorem)**.** *If $C = \bigoplus C_n$ is a chain complex and G is an Abelian group, then the cohomology groups $H^n(C; G)$ of the cochain complex $\mathrm{Hom}(C_n, G)$ are determined by the split exact sequences*

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(C; G) \to \mathrm{Hom}(H_n(C), G) \to 0.$$

> **Remark 1.7.** In particular, cohomology groups are determined by homology. $\mathrm{Ext}(H_n(C), G)$ is often easy to compute using the properties since $H_{n-1}(C_n(X)) = Z^r \oplus T$, where T is the torsion.

Proposition 1.8. *Suppose $f : C_* \to D_*$ is a chain map between chain complexes. Then, f induces a map $f^\# : D^* \to C^*$ by $f^\#(\alpha)(c) = \alpha(f(c))$ for all $\alpha \in \mathrm{Hom}(D, \mathbb{Z})$ and $c \in C$, which is a cochain map. In particular, it induces a map $f^* : H^*(D) \to H^*(C)$ sometimes denoted functorially by $H^*(f)$.*

2 Cup Product

Definition 2.1 (Cup Product)**.** For $\varphi \in C^k(X)$, $\psi \in C^\ell(X)$, define $\varphi \cup \psi \in C^{k+\ell}(X)$ as the cochain whose value on a singular simplex $C_{k+\ell}(X) \ni \sigma : [v_0, \ldots, v_{k+\ell}] \to X$ is given by

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0,\ldots,v_k]}) \cdot \psi(\sigma|_{[v_k,\ldots,v_{k+\ell}]}).$$

Lemma 2.2. *$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$ for $\varphi \in C^k(X)$, $\psi \in C^\ell(X)$.*

> **Remark 2.3.** This gives a differential-graded algebra structure on $C^*(X)$.

Corollary 2.4. *If φ, ψ are cocycles, then $\varphi \cup \psi$ is a cocycle.*

Corollary 2.5. *If φ is a coboundary and ψ is a cocycle(or vice versa), then $\varphi \cup \psi$ is a coboundary.*

Corollary 2.6. *The cup product induces a map on cohomologies:*

$$\cup : H^k(X;\mathbb{Z}) \times H^\ell(X;\mathbb{Z}) \to H^{k+\ell}(X;\mathbb{Z}).$$

2.1 Cohomology Ring

Proposition 2.7. *There exists $\mathbf{1} \in C^0(X;\mathbb{Z})$ such that for all $\sigma \in C_0(X;\mathbb{Z})$, we have $\mathbf{1}(\sigma) = 1 \in \mathbb{Z}$.*

Theorem 2.8 (Cohomology Ring)**.** *$H^*(X;\mathbb{Z}) = \bigoplus_{n=0}^\infty H^n(X;\mathbb{Z})$ is a unital skew-commutative ring where $\mathbf{1}$ is the unit and the product is $\cup$. Namely, $x \cup y = (-1)^{k\ell} y \cup x$, where $x \in H^k(X;\mathbb{Z}), y \in H^\ell(X;\mathbb{Z})$.*

2.2 Spaces with Polynomial Cohomology

Theorem 2.9. *$H^*(\mathbb{RP}^n;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ and $H^*(\mathbb{RP}^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]$, where $|\alpha| = 1$ is a generator of $H^*(\mathbb{RP}^1;\mathbb{Z}_2)$.*

Theorem 2.10. *$H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[g]/(g)$ and $H^*(\mathbb{CP}^\infty;\mathbb{Z}) = \mathbb{Z}[g]$, where g is a generator of $H^2(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$.*

> **Remark 2.11.** This is canonical by the choice of orientation given by the complex structure.

3 Poincaré Duality

3.1 Orientation

We assume familiarity with smooth manifolds.

Definition 3.1. A local orientation of a topological manifold M is a choice of generator $\mu_n \in H_n(M, M \setminus \{m\}; \mathbb{Z}) \cong \mathbb{Z}$.

Definition 3.2. An orientation on M is a choice of local orientations $M \ni m \mapsto \mu_m$ such that for every $m \in M$, there exists a neighborhood $U \ni m$ and an element $\mu_U \in H_n(M, M \setminus U; \mathbb{Z})$ such that the embedding $(M, M \setminus U) \to (M, M \setminus \{p\})$ induces a map sending $\mu_U \mapsto \mu_p$ for all $p \in U$.

Fact 3.3. Not every manifold M is orientable, but every manifold admits a double cover $\overline{M}$ which is orientable.

Definition 3.4. Let $\overline{M} := \{\mu_m : m \in M, \mu_m$ is a local orientation generating $H_n(M, M \setminus \{m\}; \mathbb{Z})\}$. The basis of the topology is as follows: for an open ball $B \subset \varphi_\alpha(\mathbb{R}^n) \subset M$ and $\mu_B \in H_n(M, M \setminus B)$, let $U(\mu_B) := \{$restrictions of μ_B to $M, M \setminus \{p\} \, \forall p \in B\}$. This gives a $2:1$ continuous covering map $\overline{M} \to M$ by $\mu_m \mapsto m$.

Lemma 3.5. *M is orientable if and only if $\overline{M}$ has two components.*

3.2 Cap Product

Definition 3.6. Assume that $k \geq \ell$. The cap product $\cap : C_k(X; \mathbb{Z}) \times C^\ell(X; \mathbb{Z}) \to C_{k-\ell}(X; \mathbb{Z})$ is defined as follows: let $\sigma \in C_k(X; \mathbb{Z})$ and $\varphi \in C^\ell(X; \mathbb{Z})$, then the image is given by

$$\sigma \cap \varphi = \varphi(\sigma|_{[v_0, \ldots, v_\ell]}) \sigma|_{[v_\ell, \ldots, v_k]}.$$

Proposition 3.7. $\partial(\sigma \cap \varphi) = (-1)^\ell (\partial\sigma \cap \varphi - \sigma \cap \delta\varphi)$.

Corollary 3.8. *$\cap$ induces a well defined map $H_k(X; \mathbb{Z}) \times H^\ell(X; \mathbb{Z}) \to H_{k-\ell}(X; \mathbb{Z})$.*

3.3 The Duality Theorems

Proposition 3.9. *If M is a closed topolpogical manifold, then orientability is equivalent to $H_n(M; \mathbb{Z}) = \mathbb{Z}$ and a choice of such an isomorphism corresponds to an orientation, i. e. a choice of fundamental cycle $[M] \in H_n(M; \mathbb{Z})$.*

Theorem 3.10 (Poincaré Duality)**.** *Suppose that M^n is a closed topological manifold which is oriented with the fundamental class $[M] \in H_n(M; \mathbb{Z})$. Then the map $D : H^k(M; \mathbb{Z}) \to H_{n-k}(M; \mathbb{Z})$ defined by $\alpha \mapsto [M] \cap \alpha$ is an isomorphism for all k.*